SIAM J. MATH. ANAL.
Vol. 44, No. 3, pp. 1340–1358

GLOBAL BIFURCATION OF SOLUTIONS FOR CRIME MODELING EQUATIONS*

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Abstract. We study pattern formation in a quasi-linear system of two elliptic equations that was developed by Short et al. [Math. Models Methods Appl. Sci., 18 (2008), pp. 1249–1267] as a model for residential burglary. That model is based on the observation that the rate of burglaries of houses that have been burglarized recently and their close neighbors is typically higher than the average rate in the larger community, which creates hotspots for burglary. The patterns generated by the model describe the location of those hotspots. We prove that the system supports global bifurcation of spatially varying solutions from the spatially constant equilibrium, leading to the formation of spatial patterns. The analysis is based on recent results on global bifurcation in quasi-linear elliptic systems derived by Shi and Wang [J. Differential Equations, 7 (2009), pp. 2788–2812]. We show in some cases that near the bifurcation point the bifurcating spatial patterns are stable.

Key words. burglary model, quasi-linear elliptic system, pattern formation, global bifurcation

AMS subject classifications. Primary, 36J47, 36J62, 91D25, 91F99; Secondary, 35B32

DOI. 10.1137/110843356

1. Introduction.

The application of mathematics to crime modeling is a rather new topic which is receiving increasing attention. Perhaps the recent work on residential burglaries [12] is the starting point for mathematical modeling of crime. In that paper a study of the dynamics of residential burglary hotspots was undertaken. The problem of understanding hotspots arises because burglaries are often observed to be clustered in certain neighborhoods. The dynamics of hotspots were modeled first by using an agent-based statistical model based on the broken window effect and repeat or near repeat victimization sociological effects. Those terms refer to the observation that the rate of burglaries of houses that have been burglarized recently and their close neighbors is typically higher than the average rate in the larger community.

The agent-based model in [12] is based on the assumption that criminal agents are walking randomly on a two-dimensional lattice and committing burglaries when encountering an attractive opportunity. An attractiveness value is assigned to every house (point in the lattice), which measures how easily the house can be burgled without negative consequences for the criminal agent. The criminal agents, in addition to walking randomly, move toward areas of high attractiveness values. In turn, when a burglary occurs, it increases the attractiveness of the house that was burglarized and those nearby. If no additional burglaries occur, then the local attractiveness decays toward a constant baseline value. Hotspots arise from pattern formation analogous to that arising in reaction-diffusion models via Turing instabilities.

*Received by the editors August 4, 2011; accepted for publication (in revised form) January 30, 2012; published electronically May 3, 2012.
http://www.siam.org/journals/sima/44-3/84335.html

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In a second step, by taking a suitable limit of the equations for the discrete model, one of which models the attractiveness of individual houses to burglary, and the other of which models burglar movement, the authors of [12] developed a continuum model based in a system of parabolic differential equations which they complemented with periodic boundary conditions. They showed that the spatially constant equilibrium can become unstable, so that pattern formation is possible. In [11] the authors did a weakly nonlinear analysis of the equilibrium system from [12]. They performed an asymptotic analysis by using perturbation techniques where they derived amplitude equations governing the development of crime hotspot patterns in both the one dimensional and radially symmetric two-dimensional cases, and found a subcritical pitchfork bifurcation in the case of one space dimension and a transcritical bifurcation in the radial case in two dimensions. They determined that the model supports hotspots even in some parameter ranges where the spatially homogeneous equilibrium is stable. In [9] the authors obtained local existence and uniqueness for the time dependent system from [12]. They also derived a continuation argument giving conditions that would imply global existence if they could be verified. They considered a generalized version of a Keller–Segel chemotaxis model as a simplification of the model from [12] and studied it as a first step toward understanding possible conditions for global existence versus blow-up of solutions in finite time for the original model. In fact, the models in [12] and especially [9] have structures that are similar to those of chemotaxis models and their generalizations. For discussions of chemotaxis models such as the Keller–Segel model, see [4, 5, 13]. The results in [13] include a bifurcation analysis. Also, Shi and Wang treat a chemotaxis model in one space dimension as an example in [10]. The analytic approaches in those papers are similar in spirit to the one we will take but differ considerably in their details. In [7] the author proposed a modification to the model from [12] incorporating deterrence due to presence of police, obtained a new system of partial differential equations, and performed numerical experiments on it. Another class of models related to the distribution of criminal activity is derived and analyzed in [1]. That paper also gives a nicely focused background discussion of modeling criminal behavior.

The parabolic system from [12] is the starting point of this paper. In contrast to [12] our boundary conditions will be of no flux type. We will use bifurcation theory to rigorously prove that the equilibrium system does indeed support pattern formation and to characterize to some extent the nature of the bifurcating patterns. This complements the formal analysis in [11]. The bifurcation analysis is based on some recent results on global bifurcation in quasi-linear elliptic systems developed in [10]. Moreover, we show in some cases that near the bifurcation point the bifurcating spatial patterns are stable.

Thus we begin by considering the problem

$$(E)\qquad \begin{cases} \dfrac{\partial A}{\partial t} = \eta\triangle A - A + A^0 + \rho A \quad \text{in} \quad \Omega\times(0,T], \\[2ex] \dfrac{\partial \rho}{\partial t} = \nabla\cdot\left[\nabla\rho - \dfrac{2\rho}{A}\nabla A\right] - \rho A + \overline{A} - A^0 \quad \text{in} \quad \Omega\times(0,T], \\[2ex] \dfrac{\partial A}{\partial n} = 0, \quad \dfrac{\partial \rho}{\partial n} - \dfrac{2\rho}{A}\dfrac{\partial A}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\times(0,T], \end{cases}$$

where $A > 0$ is the attractiveness, $\rho > 0$ is the density of potential burglars, $\eta > 0$ is the diffusion rate of attractiveness, $A^0 > 0$ is the intrinsic (static) attractiveness,

$\overline{A} \equiv \lambda$ is the average attractiveness, and Ω is a bounded domain in $\mathbb{R}^N$ of class $C^{2+\alpha}$. Note that because of the boundary condition on A the boundary condition on ρ is equivalent to the Neumann condition $\partial\rho/\partial n = 0$ on $\partial\Omega$.

We are specifically interested in finding spatially nonconstant solutions of the equilibrium problem

$$(EQ) \qquad \begin{cases} \eta\triangle A - A + A^0 + \rho A = 0 \quad \text{in} \quad \Omega, \\ \nabla \cdot \left[\nabla\rho - \dfrac{2\rho}{A}\nabla A\right] - \rho A + \overline{A} - A^0 = 0 \quad \text{in} \quad \Omega \times (0,T], \\ \dfrac{\partial A}{\partial n} = 0, \quad \dfrac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Associated with this problem we will consider the problem

$$(LN) \qquad \begin{cases} \triangle u + \mu u = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

We will be especially interested in the case where $\mu > 0$ is a *simple* eigenvalue. Clearly μ is not the principal eigenvalue in that case.

Throughout this work we will assume the following hypotheses:

$$\overline{A} > A^0, \tag{1.1}$$

$$\eta < \frac{2}{\mu} \quad \text{and} \quad A^0 < \frac{(\eta\mu^2 - 2\mu)^2}{12\mu(\eta\mu + 1)}. \tag{1.2}$$

We note that in addition to [11] and [12] , recent related work on the subject can be found in [7] and [9].

This paper is organized as follows. In section 2 we formulate the bifurcation problem and state our first main result, which gives criteria for the bifurcation of of spatially nonconstant solutions of (EQ) from the constant solution. In section 3 we determine the locations of bifurcation points. In sections 4 and 5 we verify that the conditions needed to prove the main result are satisfied. In section 6 we state and prove our second main result, which shows that in some cases the bifurcating nonconstant solutions of (EQ) are locally stable equilibria of (E). In section 7 we briefly describe our conclusions.

2. Setting of the problem and first main result. The system of ordinary differential equations associated with the model is

$$\frac{dA}{dt} = -A + A^0 + \rho A,$$

$$\frac{d\rho}{dt} = -\rho A + \overline{A} - A^0.$$

This system has one critical point given by

$$(A,\rho) = \left(\overline{A}, 1 - \frac{A^0}{\overline{A}}\right).$$

Using this critical point we make a shift of the variables in (E) by setting $\tilde{A} = A - \overline{A}$ and $\tilde{\rho} = \rho - \overline{\rho}$, where $\overline{\rho} = 1 - \frac{A^0}{\overline{A}}$ so that the critical point is now $(\tilde{A}, \tilde{\rho}) = (0, 0)$.

Rewriting the differential equations in terms of $\tilde{A}$ and $\tilde{\rho}$, we obtain

$$(ME) \qquad \begin{cases} \dfrac{\partial \tilde{A}}{\partial t} = \eta \triangle \tilde{A} - \tilde{A} + \tilde{\rho}\tilde{A} + \overline{\rho}\tilde{A} + \overline{A}\tilde{\rho} \quad \text{in} \quad \Omega, \\ \dfrac{\partial \tilde{\rho}}{\partial t} = \nabla \cdot \left[\nabla \tilde{\rho} - \dfrac{2(\tilde{\rho} + \overline{\rho})}{\tilde{A} + \overline{A}} \nabla \tilde{A} \right] - \tilde{\rho}\tilde{A} - \overline{A}\tilde{\rho} - \overline{\rho}\tilde{A} \quad \text{in} \quad \Omega, \\ \dfrac{\partial \tilde{A}}{\partial n} = 0, \quad \dfrac{\partial \tilde{\rho}}{\partial n} - \dfrac{2(\tilde{\rho} + \overline{\rho})}{\tilde{A} + \overline{A}} \dfrac{\partial \tilde{A}}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

We notice that since $\rho = \tilde{\rho} + \overline{\rho} > 0$ and $A = \tilde{A} + \overline{A} > 0$, the second boundary condition becomes $\frac{\partial \tilde{\rho}}{\partial n} = 0$.

Thus, taking this fact into account and after dropping the tildes, problem (ME) becomes

$$(MP) \qquad \begin{cases} \dfrac{\partial A}{\partial t} = \eta \triangle A - A + \rho A + \overline{\rho} A + \overline{A}\rho \quad \text{in} \quad \Omega, \\ \dfrac{\partial \rho}{\partial t} = \nabla \cdot \left[\nabla \rho - \dfrac{2(\rho + \overline{\rho})}{A + \overline{A}} \nabla A \right] - \rho A - \overline{A}\rho - \overline{\rho} A \quad \text{in} \quad \Omega, \\ \dfrac{\partial A}{\partial n} = 0, \quad \dfrac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

In this paper we will be interested in studying stationary solutions for problem (MP). By defining

$$(2.1) \quad F(\lambda, A, \rho) = \begin{bmatrix} \eta \triangle A - A + \rho A + \left(1 - \dfrac{A^0}{\lambda}\right) A + \lambda \rho \\ \nabla \cdot \left[\nabla \rho - \dfrac{2(\rho + (1 - \frac{A^0}{\lambda}))}{A + \lambda} \nabla A \right] - \rho A - \lambda \rho - \left(1 - \dfrac{A^0}{\lambda}\right) A \end{bmatrix},$$

where we have replaced $\overline{A}$ with λ in the right-hand side of (MP) (this we will do henceforth), we obtain that the stationary solutions to this problem are the solutions to the problem given by

$$(EP) \qquad \begin{cases} F(\lambda, A, \rho) = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial A}{\partial n} = 0, \quad \dfrac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

We observe here that $(\lambda, A, \rho) = (\lambda, 0, 0)$ is a solution to this problem for any $\lambda \in \mathbb{R}$. This motivates us to find nontrivial stationary solutions as a branch of solutions bifurcating from a particular value of λ. To this end we will use the following result of Shi and Wang (see [10, Theorem 4.3]), which is a version of a well-known local bifurcation theorem due to Crandall and Rabinowitz [2]. This version of the theorem also gives conditions for global bifurcation of the type due to Rabinowitz [8] but recast in such a way that the result can be applied directly to quasi-linear elliptic systems without first converting them to fixed point equations for compact perturbations of the identity.

Let $\mathcal{Y}$ and $\mathcal{Z}$ be real Banach spaces and $\mathcal{V}$ an open set in $\mathbb{R} \times \mathcal{Y}$ such that $(\lambda_0, 0) \in \mathcal{V}$. If $\mathcal{A}$ is a linear operator, then we will denote the null space and range of $\mathcal{A}$ respectively by $\text{Ker}(\mathcal{A})$ and $R(\mathcal{A})$; dimension and codimension will be denoted by dim and codim, respectively.

THEOREM 2.1. *Let*

$$\mathcal{F} : \mathcal{V} \mapsto \mathcal{Z}$$

be such that $\mathcal{F}$ *is continuously differentiable and satisfies the following:* (a) $\mathcal{F}(\lambda, 0) = 0$ *for all* $(\lambda, 0) \in \mathcal{V}$; (b) *the partial derivative* $D\mathcal{F}_{\lambda y}(\lambda, y)$ *exists and is continuous;* (c) *for some* $(\lambda_0, 0) \in \mathcal{V}$, $R(D\mathcal{F}_y(\lambda_0, 0)$ *is closed,* $\dim \text{Ker}(D\mathcal{F}_y(\lambda_0, 0)) = 1$, *and* $\text{codim}\, R(D\mathcal{F}_y(\lambda_0, 0)) = 1$. *Suppose further that* (d) $D\mathcal{F}_{\lambda y}(\lambda_0, 0)y_0 \notin R((D\mathcal{F}_y(\lambda_0, 0))$, *where* y_0 *spans* $\text{Ker}(D\mathcal{F}_y(\lambda_0, 0))$.

Let $\mathcal{W} \subset \mathcal{Y}$ *be any closed complement of the one-dimensional space spanned by* y_0. *Then there exist an open interval* I_0 *containing* 0 *and continuously differentiable functions* $\lambda : I_0 \mapsto \mathbb{R}$ *and* $\xi : I_0 \mapsto \mathcal{W}$ *with* $\lambda(0) = \lambda_0$, $\xi(0) = 0$, *such that*

$$\mathcal{F}(\lambda(s), sy_0 + s\xi(s)) = 0 \quad \textit{for } s \in I_0.$$

In addition the entire solution set for $\mathcal{F}(\lambda, y) = 0$ *in any sufficiently small neighborhood of* $(\lambda, 0)$ *in* $\mathcal{V}$ *consists of the line* $(\lambda, 0)$ *and the curve* $(\lambda(s), sy_0 + s\xi(s))$. *Furthermore, if* (e) $D\mathcal{F}_y(\lambda, y)$ *is a Fredholm operator for all* $(\lambda, y) \in \mathcal{V}$, *then the curve* $(\lambda(s), sy_0 + s\xi(s))$ *is contained in* C, *which is a connected component of* $\overline{S}$, *where* $S = \{(\lambda, y) \in \mathcal{V} : \mathcal{F}(\lambda, y) = 0, y \neq 0\}$ *and either* C *is not compact in* $\mathcal{V}$ *or* C *contains a point* $(\lambda^*, 0)$ *with* $\lambda^* \neq \lambda_0$.

In what follows we will prove that under certain conditions on the parameters of the problem all the conditions of this theorem are satisfied for the problem (EP).

To this end we specify that in Theorem 2.1 $\mathcal{F} = F$ as defined in (2.1), $\mathcal{Y} = Y$, $\mathcal{V} = V$, and $\mathcal{Z} = Z$, where Y, V, and Z are respectively defined by

$$Y = \left\{ (A, \rho) \in \left[W^{2,p}(\Omega) \right]^2 \,\middle|\, \frac{\partial A}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \right\},$$

$$V = \left\{ (\lambda, A, \rho) \in \mathbb{R} \times Y \,\middle|\, \lambda > A^0 + \varepsilon, A > -\lambda + \varepsilon, \rho > \frac{A^0}{\lambda} - 1 \right\}, \qquad Z = \left[L^p(\Omega) \right]^2,$$

where $\varepsilon > 0$ is small and $p > n$.

Recall that $W^{2,p}(\Omega)$ embeds in $C^{1+\alpha}(\overline{\Omega})$ if $p > n$, so the boundary conditions in the definition of Y make sense. Note that for $\lambda > A^0 + \varepsilon$ we have $A > -\lambda + \varepsilon$ provided that $A > -A^0$, so that the definition of V does not unreasonably restrict A. The restriction $\rho > \frac{A^0}{\lambda} - 1$ is imposed because we identify λ with $\overline{A}$, and in the application the quantity $\rho + 1 - \frac{A^0}{A}$ represents the density of criminals and so should be positive. (Recall that we have shifted the variables from the original model.) The restriction on ρ is not necessary for the mathematical analysis.

All of the parts of Theorem (2.1) except the global bifurcation results on C are essentially the same as in the original formulation of the local bifurcation theorem of

Crandall and Rabinowitz [3]. Thus, in applying those parts of Theorem 2.1 to our system to obtain local bifurcations we could replace Y with

$$\left\{(A,\rho)\in\left[C^{2+\alpha}(\overline{\Omega})\right]^2 \;\middle|\; \frac{\partial A}{\partial n}=\frac{\partial\rho}{\partial n}=0 \quad \text{on} \quad \partial\Omega\right\}$$

and Z with

$$\left[C^{\alpha}(\overline{\Omega})\right]^2.$$

The only requirement would be that we work in spaces where the elliptic operator defined by $D\mathcal{F}_y(\lambda_0,0)$ is Fredholm with index zero and with zero as a simple eigenvalue. That requirement is satisfied in both Sobolev and Hölder spaces. However, the argument developed by Shi and Wang to verify hypothesis (e) in applying Theorem 2.1 to obtain global bifurcation in quasi-linear systems uses regularity results formulated in Sobolev spaces. It might be possible to formulate analogous results in Hölder spaces, but that is beyond the scope of this paper, so we set our theorem in Sobolev spaces.

The regularity results necessary to establish the Fredholm properties of $D\mathcal{F}_y$ in the setting of quasi-linear second order elliptic systems on Sobolev spaces are discussed in [10]. We will explain later why the results of [10] apply to our model, but we refer the reader to that paper for a detailed discussion of the underlying regularity theory.

We will characterize possible bifurcation points for our model in terms of eigenvalues of (LN). For a given simple eigenvalue μ of (LN), let ϕ be a normalized eigenfunction corresponding to μ. Under suitable conditions on μ and the parameters in (EP), we will show that there is a bifurcation point λ_0 for (EP) associated with μ, and that if λ_0 is such a point, then $\mathrm{Ker}(D\mathcal{F}_y(\lambda_0,0))$ is spanned by $(\lambda_0\phi,(\eta\mu+\frac{A^0}{\lambda_0})\phi)$. Thus, if we let ℓ be the linear functional on Y defined by

$$\ell(p,q)=\int_\Omega\left[\lambda_0 p+\left(\eta\mu+\frac{A^0}{\lambda_0}\right)q\right]\phi, \tag{2.2}$$

then

$$\mathcal{W}=W=\{(p,q)\in Y\mid \ell(p,q)=0\}$$

is a closed complement of $\mathrm{Ker}(D\mathcal{F}_y(\lambda_0,0))$ in $\mathcal{Y}$.

We will prove the validity of the following theorem, which is our first main result in this paper. Recall that we have identified the bifurcation parameter λ with the parameter $\overline{A}$ in the original model.

THEOREM 2.2 (first main result). *Suppose that $\mu>0$ is a simple eigenvalue of (LN) and that conditions (1.1) and (1.2) are satisfied. Let $\lambda_0>A^0$ be a solution of*

$$3\frac{A^0}{\lambda^2}+\frac{(\eta\mu-2)}{\lambda}+\eta+\frac{1}{\mu}=0$$

such that $\frac{\lambda_0}{\eta\mu}$ is not an eigenvalue of (LN). Then a branch of spatially nonconstant solutions of (2.1) bifurcates from the equilibrium $(\overline{A},1-\frac{A^0}{\overline{A}})$ at $\overline{A}=\lambda_0$. In a neighborhood of the bifurcation point, the bifurcating branch can be parameterized as $(\overline{A},A,\rho)=(\overline{A}(s),\overline{A}(s)+s\lambda_0\phi+s\xi_1(s),(1-\frac{A^0}{\overline{A}(s)})+s(\eta\mu+\frac{A^0}{\lambda_0})\phi+s\xi_2(s))$, where ϕ is a normalized eigenfunction of (LN) and $(\xi_1,\xi_2)\in W$, and where $\overline{A}(0)=\lambda_0$

and $(\xi_1(0), \xi_2(0)) = (0,0)$. *Furthermore, the bifurcating branch is part of a connected component* C_0 *of the set* $\overline{S}$, *where* $S = \{(\overline{A}, A, \rho) : (\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) \in V, F(\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) = 0, (A, \rho) \neq (\overline{A}, 1 - \frac{A^0}{\overline{A}})\}$, *and* C_0 *either extends to infinity in* $\overline{A}, A$, *or* ρ, *or contains a point where* $(\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) \in \partial V$, *or contains a point* $(\lambda^*, \overline{A}, 1 - \frac{A^0}{\overline{A}})$ *with* $\lambda^* \neq \lambda_0$.

Remark 2.3. If $\frac{\lambda_0}{\eta\mu}$ is not an eigenvalue of (LN), then, in particular,

$$\mu \neq \frac{\lambda_0}{\eta\mu}. \tag{2.3}$$

Remark 2.4. If any closed bounded subset of the set of solutions to $\mathcal{F}(\lambda, y) = 0$ in $\mathcal{V}$ is compact and the alternative that occurs in Theorem 2.1 is that $\mathcal{C}$ is not compact in $\mathcal{V}$, then $\mathcal{C}$ must either be unbounded or must leave $\mathcal{V}$, but in the latter case since $\mathcal{C}$ is connected it must contain a point in $\partial\mathcal{V}$. In our application of Theorem 2.1 to prove Theorem 2.2, that turns out to be the case.

In this respect we have the following lemma.

LEMMA 2.5. *Any closed bounded subset of the solution set of* (*EP*) *in* V *is compact.*

Proof. Since $p > n$, it follows from the Sobolev embedding theorem that any closed, bounded set of $W^{2,p}(\Omega)$ is also closed and bounded in $C^{1+\alpha}(\overline{\Omega})$. Thus, for (A, ρ) restricted to a closed bounded subset of the solution set of (EP) in V we have that A and ρ are uniformly bounded in $C^{1+\alpha}(\overline{\Omega})$. Since

$$\eta \triangle A = A - \rho A - \left(1 - \frac{A^0}{\lambda}\right) A - \lambda\rho$$

and A satisfies a homogeneous Neumann boundary condition, it follows from Theorems 3.1 and 3.2 and inequality (3.7) of [6, Chapter 3, section 3] that A is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$. We have that ρ satisfies a homogeneous Neumann boundary condition and the equation

$$\triangle\rho - \frac{2\nabla\rho \cdot \nabla A}{A + \lambda} = \nabla \cdot \left[\frac{2(1 - \frac{A^0}{\lambda})}{A + \lambda}\nabla A\right] + \rho A + \lambda\rho + \left(1 - \frac{A^0}{\lambda}\right) A. \tag{2.4}$$

Since A is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$, the coefficients and the right-hand side of (2.4) are all uniformly bounded in $C^{\alpha}(\overline{\Omega})$, so that again the results of [6, Chapter 3, section 3] imply that ρ is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$. Since $[C^{2+\alpha}(\overline{\Omega})]^2$ embeds compactly in $[C^2(\overline{\Omega})]^2$, which in turn embeds in $[W^{2,p}(\Omega)]^2$, the conclusion of the lemma follows.

Finally recall that the variables in (EP) are shifted from those of the original system (EQ) to move the equilibrium $(\overline{A}, 1 - \frac{A^0}{\overline{A}})$ to $(0,0)$. We will apply Theorem 2.1 to (EP) and then return to the original variables to obtain Theorem 2.2. Thus we will first prove the following result.

THEOREM 2.6 (first bifurcation result). *Under the conditions of Theorem* 2.2, *a branch of spatially nonconstant solutions of* (*EQ*) *bifurcates from the equilibrium* $(0,0)$ *at* $\lambda = \lambda_0$. *In a neighborhood of the bifurcation point, the bifurcating branch can be parameterized as*

$$(\lambda, A, \rho) = \left(\lambda(s), s\lambda_0\phi + s\xi_1(s), s\left(\eta\mu + \frac{A^0}{\lambda_0}\right)\phi + s\xi_2(s)\right),$$

where $\lambda(0) = \lambda_0$, *and where* $(\xi_1, \xi_2) \in W$ *with* $(\xi_1(0), \xi_2(0)) = (0,0)$.

Furthermore, the bifurcating branch is part of a connected component C_0 of the set $\overline{S}$, where

$$S = \{(\lambda, A, \rho) \in V : F(\lambda, A, \rho) = 0, (A, \rho) \neq (0,0)\},$$

and C_0 either extends to infinity in λ, A, or ρ, or contains a point where $(\lambda, A, \rho) \in \partial V$, or contains a point $(\lambda^*, 0, 0)$ with $\lambda^* \neq \lambda_0$.

3. Searching for possible bifurcation points. We begin by noticing that F as in (2.1) clearly defines a differentiable mapping from V to Z.

By linearizing $F(\lambda, A, \rho)$ with respect to (A, ρ), we find

$$DF_{(A,\rho)}(\lambda, A, \rho)(u, v) \tag{3.1}$$
$$= \begin{bmatrix} \eta\Delta u - u + \rho u + Av + \left(1 - \frac{A^0}{\lambda}\right) u + \lambda v \\ \left(\nabla\left[\nabla v - \frac{2v}{A+\lambda}\nabla A - 2\left(\rho + 1 - \frac{A^0}{\lambda}\right)\left(\frac{\nabla u}{A+\lambda} - \frac{u\nabla A}{(A+\lambda)^2}\right)\right]\right. \\ \left. -\rho u - Av - \left(1 - \frac{A^0}{\lambda}\right) u - \lambda v\right) \end{bmatrix}.$$

It is clear that $DF_{(A,\rho)}(\lambda, A, \rho)$ is a bounded operator from Y to Z that is continuous and differentiable with respect to λ, A, and ρ in V. Evaluating at $(\lambda, 0, 0)$, after some computations, we obtain

$$DF_{(A,\rho)}(\lambda, 0, 0)(u, v) = \begin{bmatrix} \eta\Delta u - \frac{A^0}{\lambda} u + \lambda v \\ \Delta v - \frac{2}{\lambda}\left(1 - \frac{A^0}{\lambda}\right)\Delta u - \left(1 - \frac{A^0}{\lambda}\right) u - \lambda v \end{bmatrix}. \tag{3.2}$$

It is straightforward to compute the linearization $DF_\lambda(\lambda, A, \rho)$ of F with respect to λ, and it is clear that $DF_\lambda(\lambda, A, \rho)$ is continuous and differentiable with respect to λ, A, and ρ in V. Since we do not need to use $DF_\lambda(\lambda, A, \rho)$ directly, we do not show it here.

For bifurcation from a particular value of λ, we need the implicit function theorem to fail. Thus we need the mapping $DF_{(A,\rho)}(\lambda, 0, 0)$ to have a nontrivial kernel for that value of λ; in other words this mapping must have zero as an eigenvalue.

To identify these λ's, we search for nontrivial solutions pairs (u, v) for the problem

$$(LE) \qquad \begin{cases} \eta\Delta u - \frac{A^0}{\lambda} u + \lambda v = 0 \quad \text{in} \quad \Omega, \\ \frac{2}{\lambda}\left(1 - \frac{A^0}{\lambda}\right)\Delta u - \Delta v + \left(1 - \frac{A^0}{\lambda}\right) u + \lambda v = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

It follows from classical results on partial differential equations that for any pair of functions $(u, v) \in Y$ both u and v can be expanded as series of mutually orthogonal eigenfunctions of (LN) multiplied by constant vectors. If (u, v) is nonzero, then the coefficient of at least one of those eigenfunctions in at least one of those expansions must be a nonzero vector. Suppose that ϕ is such an eigenfunction, corresponding to an eigenvalue μ, and normalized by

$$\int_\Omega \phi^2 = 1.$$

Define U and V by

$$U = \int_\Omega u\phi \quad \text{and} \quad V = \int_\Omega v\phi. \tag{3.3}$$

Multiplying the first two equations in (LE) by ϕ and integrating over Ω, using the boundary conditions, and taking into account that

$$\int_\Omega \phi\Delta u = -\mu \int_\Omega \phi u = -\mu U \quad \text{and} \quad \int_\Omega \phi\Delta v = -\mu \int_\Omega \phi w = -\mu V, \tag{3.4}$$

we are led to the following linear system for U and V:

$$\begin{bmatrix} -\eta\mu - \dfrac{A^0}{\lambda} & \lambda \\ \dfrac{2\mu}{\lambda}\left(1 - \dfrac{A^0}{\lambda}\right) - \left(1 - \dfrac{A^0}{\lambda}\right) & -\lambda - \mu \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = 0. \tag{3.5}$$

To have nontrivial solutions (nontrivial U, V), we set

$$\begin{vmatrix} -\eta\mu - \dfrac{A^0}{\lambda} & \lambda \\ \dfrac{2\mu}{\lambda}\left(1 - \dfrac{A^0}{\lambda}\right) - \left(1 - \dfrac{A^0}{\lambda}\right) & -\lambda - \mu \end{vmatrix} = 0, \tag{3.6}$$

which gives the following relationship between the parameters involved:

$$3\frac{A^0}{\lambda^2} + \frac{(\eta\mu - 2)}{\lambda} + \eta + \frac{1}{\mu} = 0. \tag{3.7}$$

Setting $\Lambda = \frac{1}{\lambda}$, this relationship can be seen as a quadratic equation for Λ, namely

$$p(\Lambda) := 3A^0\mu\Lambda^2 + (\eta\mu - 2)\mu\Lambda + \eta\mu + 1 = 0. \tag{3.8}$$

If this equation is satisfied, then (3.5) will have a nontrivial solution (U, V). In that case a nontrivial solution (u, v) of (LE) is given by

$$u = U\phi \quad \text{and} \quad v = V\phi. \tag{3.9}$$

We will see that for those values of λ where the remaining hypotheses of Theorem 2.1 are satisfied all solutions of (LE) must be of that form.

A simple calculation shows that $p(\Lambda)$ has a positive root if conditions (1.2) are satisfied. Actually, under these conditions both roots of $p(\Lambda)$ will be positive. Notice that the second inequality in (1.2) can be set as an equality and still $p(\Lambda)$ will have positive roots. We assume the strict inequality for reasons that will become clear rather immediately.

From (3.8), $p(\Lambda)$ has a double root if and only if

$$(\eta\mu^2 - 2\mu)^2 - 4(3A^0\mu)(\eta\mu + 1) = 0.$$

This is not possible by the second condition of (1.2), and thus, under (1.2), $p(\Lambda)$ has two simple positive roots.

On the other hand, by evaluating the first derivative of $p(\Lambda)$, we find

$$p'(\Lambda) = (6A^0\Lambda + \eta\mu - 2)\mu = \left(\frac{6A^0}{\lambda} + \eta\mu - 2\right)\mu,$$

where we have replaced Λ by $\frac{1}{\lambda}$. Since at a root of $p(\Lambda)$ necessarily $p'(\Lambda) \neq 0$, it must be that at that root

$$\frac{6A^0}{\lambda} + \eta\mu - 2 \neq 0. \tag{3.10}$$

This condition will appear again later in a crucial form.

In summary, under conditions (1.2) we can determine a pair of positive roots of $p(\lambda)$. For each of these roots the pair (u, v) given by (3.9), where (U, V) satisfies (3.5) and ϕ is an eigenfunction to problem (LN), satisfies (LE).

Furthermore, for each root λ, $(\lambda, 0, 0)$ will be a possible bifurcation point for problem (EP). To actually have a bifurcation point we need to verify that conditions (c) and (d) of Theorem 2.1 are satisfied for that value of λ. This we will do in the next two sections.

4. (c) and (e) of Theorem 2.1. In this section we seek conditions for the validity of (c) and (e) of Theorem 2.1. Suppose that $\lambda = \lambda_0$ is a positive root of $p(\lambda)$ as above. To verify (c) we begin by looking for conditions among parameters for $\mathrm{Ker}(DF_{(A,\rho)}(\lambda_0, 0, 0))$ to be one-dimensional, in other words, for the dimension of the linear space of solutions of problem (LE) to be one-dimensional.

By multiplying the first equation of (LE) (with $\lambda = \lambda_0$) by $-\frac{2\eta}{\lambda_0}(1 - \frac{A^0}{\lambda_0})$ and adding to the second (with $\lambda = \lambda_0$), we obtain that problem (LE) can be recast as the system

$$\begin{bmatrix} \triangle u \\ \triangle v \end{bmatrix} + M \begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{4.2}$$

where M is the matrix

$$M = \begin{bmatrix} -\dfrac{A^0}{\eta\lambda_0} & \dfrac{\lambda_0}{\eta} \\ -\left(\dfrac{2A^0}{\eta\lambda_0^2} + 1\right)\left(1 - \dfrac{A^0}{\lambda_0}\right) & \dfrac{2}{\eta}\left(1 - \dfrac{A^0}{\lambda_0}\right) - \lambda_0 \end{bmatrix}. \tag{4.3}$$

Using (2.3) and (3.7), we have that M has the two different positive real eigenvalues $\sigma_1 = \mu$ and $\sigma_2 = \frac{\lambda_0}{\mu\eta}$, where we recall that μ is a simple eigenvalue of problem (LN).

Denoting by K the matrix whose columns are the eigenvectors corresponding to σ_1 and σ_2, respectively, we find that

$$K^{-1}MK = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}.$$

Thus by setting

$$\begin{bmatrix} w \\ y \end{bmatrix} = K^{-1} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{4.4}$$

we have that (4.1) can be uncoupled, and we obtain

$$\begin{bmatrix} \triangle w \\ \triangle y \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix} = 0 \quad \text{in} \quad \Omega, \tag{4.5}$$

$$\frac{\partial w}{\partial n} = 0, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{4.6}$$

Thus, by the hypothesis that $\frac{\lambda_0}{\mu\eta}$ is not an eigenvalue of problem (LN), we have $y = 0$, and since $\sigma_1 = \mu$ is an eigenvalue of (LN), we obtain that $w = C\phi$, where C is a constant. From (4.4), it follows that

$$u = C_1\phi, \quad v = C_2\phi, \tag{4.7}$$

where C_1, C_2 are constants and hence that $\mathrm{Ker}(DF_{(A,\rho)}(\lambda_0, 0, 0))$ is one-dimensional. The constants C_1 and C_2, respectively, are the same as U and V in (3.3).

We notice that by the first equation of (3.5),

$$-\left(\eta\mu + \frac{A^0}{\lambda_0}\right) C_1 + \lambda_0 C_2 = 0. \tag{4.8}$$

We now turn to condition (e) in Theorem 2.1. Following the discussion of example 4.2 in [10], we can write (EP) as

$$-A_1(\lambda, A, \rho)[\triangle A, \triangle \rho]^t + g(\lambda, A, \rho, \nabla A, \nabla \rho) = 0, \tag{4.9}$$

where

$$A_1(\lambda, A, \rho) = \begin{bmatrix} \eta & 0 \\ -\dfrac{2(\rho + (1 - \frac{A^0}{\lambda}))}{A + \lambda} & 1 \end{bmatrix}. \tag{4.10}$$

Similarly, we have

$$DF_{(A,\rho)}(\lambda, A, \rho)(u, v) = A_1(\lambda, A, \rho)[\triangle u, \triangle v]^t + \text{lower order terms in } (u, v).$$

Thus, A_1 defines the principal part of the elliptic operators in (EP) and $DF_{(A,\rho)}$ (λ, A, ρ). The matrix A_1 has the structure that is needed to satisfy [10, Remark 2.5.5, case 3]. It follows that Definitions 2.1–2.4 of [10] are met provided

$$\begin{vmatrix} \eta + \sigma & 0 \\ -\dfrac{2(\rho + (1 - \frac{A^0}{\lambda}))}{A + \lambda} & 1 + \sigma \end{vmatrix} \neq 0 \tag{4.11}$$

for $(\lambda, A, \rho) \in V$, where $\sigma \in \mathbb{C}$ and $\sigma = 0$ or $arg\,\sigma \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since the determinant in (4.11) is equal to $(\eta+\sigma)(1+\sigma)$, (4.11) is indeed satisfied for such σ. Thus, Agmon's condition (see [10, section 2]) is satisfied, so it follows from Corollary 2.11 of [10] that $DF_{(A,\rho)}(\lambda, A, \rho)$ is Fredholm of index 0 for $(\lambda, A, \rho) \in V$ as needed. (See also the arguments used in [10, Example 4.2], specifically Theorem 3.3 and Remark 3.4.)

Since we have already established that $\mathrm{Ker}(DF_{(A,\rho)}(\lambda_0, 0, 0))$ is one-dimensional and that $DF_{(A,\rho)}(\lambda, A, \rho)$ is Fredholm of index 0 for $(\lambda, A, \rho) \in V$, it follows that codim $R(DF_y(\lambda_0, 0)) = 1$. That observation completes the verification of condition (c) of Theorem 2.1.

5. (d) of Theorem 2.1.

Let us define

$$y_0 = \left(\lambda_0\phi, \left(\eta\mu + \frac{A^0}{\lambda_0}\right)\phi\right).$$

Then, from (4.8), it is clear that y_0 is a generator of $\mathrm{Ker}(DF_{(A,\rho)}(\lambda_0, 0, 0))$. From (3.1), we obtain

(5.1)
$$DF_{\lambda(A,\rho)}(\lambda, A, \rho)(u, v) = \begin{bmatrix} \dfrac{A^0}{\lambda^2}u + v \\ \left(\nabla\left[\dfrac{2v}{(A+\lambda)^2}\nabla A - 2\dfrac{A^0}{\lambda^2}\left(\dfrac{\nabla u}{A+\lambda} - \dfrac{u\nabla A}{(A+\lambda)^2}\right)\right.\right. \\ \left.-2\left(\rho + 1 - \dfrac{A^0}{\lambda}\right)\left(\dfrac{-\nabla u}{(A+\lambda)^2} + \dfrac{2u\nabla A}{(A+\lambda)^3}\right)\right] \\ \left. -\dfrac{A^0}{\lambda^2}u - v\right) \end{bmatrix}.$$

It is clear that $DF_{\lambda(A,\rho)}(\lambda, A, \rho)$ is continuous on V. Evaluating at $(\lambda_0, 0, 0)$ yields

$$DF_{\lambda(A,\rho)}(\lambda_0, 0, 0)(u, v) = \begin{bmatrix} \dfrac{A^0}{\lambda_0^2}u + v \\ \dfrac{2}{\lambda_0^2}\left(1 - \dfrac{2A^0}{\lambda_0}\right)\Delta u - \dfrac{A^0}{\lambda_0^2}u - v \end{bmatrix}. \tag{5.2}$$

Finally

$$DF_{\lambda(A,\rho)}(\lambda_0, 0, 0)y_0 = \begin{bmatrix} \dfrac{2A^0}{\lambda_0} + \eta\mu \\ -\dfrac{2}{\lambda_0}\left(1 - \dfrac{2A^0}{\lambda_0}\right)\mu - 2\dfrac{A^0}{\lambda_0} - \eta\mu \end{bmatrix}\phi. \tag{5.3}$$

Next we want to show that

$$DF_{\lambda(A,\rho)}(\lambda_0, 0, 0)y_0 \notin R(DF_{(A,\rho)}(\lambda_0, 0, 0)). \tag{5.4}$$

We argue by contradiction and suppose that there are u, v such that

$$DF_{(A,\rho)}(\lambda_0, 0, 0)(u, v) = DF_{\lambda(A,\rho)}(\lambda_0, 0, 0)y_0, \tag{5.5}$$

which is equivalent to

$$(S_1)\quad \begin{cases} \eta\Delta u - \dfrac{A^0}{\lambda_0}u + \lambda_0 v = \left(\dfrac{2A^0}{\lambda_0} + \eta\mu\right)\phi \quad \text{in} \quad \Omega, \\ \Delta v - \dfrac{2}{\lambda_0}\left(1 - \dfrac{A^0}{\lambda_0}\right)\Delta u - \left(1 - \dfrac{A^0}{\lambda_0}\right)u - \lambda_0 v \\ \quad = -\left(\dfrac{2}{\lambda_0}\left(1 - \dfrac{2A^0}{\lambda_0}\right)\mu + 2\dfrac{A^0}{\lambda_0} + \eta\mu\right)\phi \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Using U, V as in (3.3), we can see that by multiplying the first two equations in (S_1) by ϕ, integrating, using the boundary conditions, and taking into account (3.4) we obtain the following algebraic system for U, V:

(5.6)

$$\begin{bmatrix} -\left(\dfrac{A^0}{\lambda_0} + \eta\mu\right) & \lambda_0 \\ \left(\dfrac{2\mu}{\lambda_0} - 1\right)\left(1 - \dfrac{A^0}{\lambda_0}\right) & -(\lambda_0 + \mu) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \dfrac{2A^0}{\lambda_0} + \eta\mu \\ -\dfrac{2\mu}{\lambda_0}\left(1 - \dfrac{2A^0}{\lambda_0}\right) - 2\dfrac{A^0}{\lambda_0} - \eta\mu \end{bmatrix}.$$

By (3.6) the determinant of the matrix of coefficients on the left-hand side of this system is zero. Then for this system to have a solution it must be that

$$-(\lambda_0 + \mu)\left(\frac{2A^0}{\lambda_0} + \eta\mu\right) + \lambda_0\left(\frac{2\mu}{\lambda_0}\left(1 - \frac{2A^0}{\lambda_0}\right) + 2\frac{A^0}{\lambda_0} + \eta\mu\right) = 0,$$

which after some simplifications yields

$$\mu\left(\eta\mu - 2 + \frac{6A^0}{\lambda_0}\right) = 0,$$

which is not possible by (3.10). Thus (5.4) holds.

Finally, since all the conditions of Theorem 2.1 are satisfied we have proved the validity of Theorem 2.6 and hence the validity of our first main result, Theorem 2.2.

6. Stability analysis of the bifurcation branches and second main result. Our second main result concerns the stability or not of the spatially inhomogeneous patterns of attractiveness and density of burglars that arise from the homogeneous patterns $(\overline{A}, 1 - \frac{A^0}{\overline{A}})$ at $\overline{A} = \lambda_0$. Here, stability refers to the stability of these inhomogeneous patterns viewed as equilibria to (E). To this end we will employ the classical results of Crandall and Rabinowitz [3] on bifurcation, perturbation of simple eigenvalues, and linearized stability in conjunction with an analysis of the spectrum of the system.

Recall that Corollary 1.13 in [3] implies that if $\lambda'(0) \neq 0$, where $\lambda(s) = \overline{A}(s)$ as in Theorem 2.2, then the eigenvalue $\sigma(s)$ of the linearization of (2.1) at the point corresponding to

$$\left(\lambda(s), s\lambda_0\phi + s\xi_1(s), s\left(\eta\mu + \frac{A^0}{\lambda_0}\right)\phi + s\xi_2(s)\right)$$

changes sign at $s = 0$. In particular, for s on one side of zero, $\sigma(s) < 0$. (Note that in [3], $\mu(s)$ is used to denote these eigenvalues. In our work, we have used μ to designate eigenvalues of $-\Delta$ on the underlying spatial domain Ω, subject to homogeneous Neumann boundary conditions.)

Consequently, it is the case that if $\sigma(0)$ is the largest eigenvalue of the linearization of (2.1) at $(\lambda_0, 0, 0)$, then $\sigma(s) < 0$ is the largest eigenvalue of the linearization of (2.1) at $(\lambda(s), A(s), \rho(s))$ for either $s > 0$ and small or $s < 0$ and small. As a result, in such a case, some of the inhomogeneous patterns that arise at $\lambda = \lambda_0$ are in fact stable when viewed as equilibrium solutions to (E). Now the eigenvalues of the linearization of (2.1) depend on the eigenvalues μ of $-\Delta$ on Ω subjected to homogeneous Neumann boundary data. Moreover, different values of μ may be associated with the eigenvalues for a particular linearization of (2.1). Consequently, obtaining that $0 = \sigma(0)$ is the largest eigenvalue of the linearization of (2.1) at $(\lambda_0, 0, 0)$ requires a closer examination of its spectral properties viz-a-viz the eigenvalues μ.

We have the following result.

THEOREM 6.1. *Suppose that μ is the eigenvalue of $-\Delta$ appearing in Theorem* 2.2, *with normalized eigenfunction ϕ. Suppose further that the hypotheses of Theorem* 2.2 *are satisfied. If $\mu \neq \lambda_0$, $\int_\Omega \phi^3 dx \neq 0$, and there is no eigenvalue of $-\Delta$ other than μ in the closed interval with endpoints μ and $\lambda_0 + \frac{\lambda_0}{\mu\eta}$, then the branch $(\overline{A}(s), A(s), \rho(s))$ of solutions of (EQ) bifurcating from the spatially constant equilibrium $(\overline{A}, \overline{A}, 1 - \frac{A^0}{\overline{A}})$ at $\overline{A} = \lambda_0$ is stable either for $s > 0$ and $|s|$ sufficiently small or for $s < 0$ and $|s|$ sufficiently small.*

Remark 6.2. By (3.7), $\lambda_0 + \frac{\lambda_0}{\mu\eta} = \frac{2}{\eta} - 3\frac{A^0}{\eta\lambda_0} - \lambda_0 - \mu$, so that the closed interval with endpoints μ and $\lambda_0 + \frac{\lambda_0}{\mu\eta}$ is contained in the interval $(0, \frac{2}{\eta})$.

Remark 6.3. In the cases where Ω is an interval or a Cartesian product of intervals, the eigenfunctions of $-\Delta$ will have odd symmetry so that $\int_\Omega \phi^3 dx = 0$. However, in a circular region, the radially symmetric eigenfunctions of the form $\phi = J_0(\sqrt{\mu}r)$ arising from Bessel functions have $\int_\Omega \phi^3 dx \neq 0$ in some cases, so our stability analysis would apply in that setting. The key idea in the proof of Theorem 6.1 is that the hypotheses imply that the bifurcation at λ_0 is transcritical, from which it follows from results of [3] that the eigenvalue of $DF_{(A,\rho)}(\lambda(s), A(s), \rho(s))$ that is zero at $s = 0$ (that is, at $(\lambda(s), A(s), \rho(s)) = (\lambda_0, 0, 0)$) changes sign as s passes thorough 0. If that eigenvalue is the largest eigenvalue of $DF_{(A,\rho)}(\lambda_0, 0, 0)$, which the hypotheses of Theorem 6.1 also imply, then it follows that the largest eigenvalue of $DF_{(A,\rho)}(\lambda(s), A(s), \rho(s))$ is negative either for $s < 0$ and $|s|$ sufficiently small or for $s > 0$ and $|s|$ sufficiently small. The conclusion that the direction of bifurcation is transverse in some radial cases but is not necessarily transverse for intervals or rectangles is consistent with the results of the asymptotic analysis in [11], which indicate that the bifurcation of spatially varying solutions is of pitchfork type if Ω is an interval but can be transcritical in the radial case. The eigenfunctions of $-\Delta$ on the disk that have nontrivial dependence on θ will have odd symmetry about some line so that $\int_\Omega \phi^3 dx = 0$ and hence will give rise to a "vertical" bifurcation. It would be of interest to determine whether the bifurcation is of pitchfork type in those cases but that is beyond the scope of this paper.

Proof of Theorem 6.1. The branch of solutions $(\overline{A}(s), A(s), \rho(s))$ of (EQ) will be stable if the corresponding branch of solutions to the translated problem (EP) are stable in (MP). This will be the case if all eigenvalues of $DF_{(A,\rho)}(\lambda(s), A(s), \rho(s))$ are negative.

From Corollary 1.13 in [3], there exist intervals I, J with $\lambda_0 \in I$, $0 \in J$ and continuously differentiable functions $\gamma : I \mapsto \mathbb{R}$, $\sigma : J \mapsto \mathbb{R}$, $(u_1, u_2) : I \mapsto Y$, $(w_1, w_2) : I \mapsto Y$, such that

$$DF_{(A,\rho)}(\lambda, 0, 0) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \gamma(\lambda) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{6.1}$$

$$DF_{(A,\rho)}(\lambda(s), A(s), \rho(s)) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \sigma(s) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{6.2}$$

such that

$$\lambda(0) = \lambda_0, \quad \gamma(\lambda_0) = \sigma(0) = 0, \quad (u_1(\lambda_0), u_2(\lambda_0)) = (w_1(0), w_2(0)) = y_0,$$

where as before $y_0 = (\lambda_0 \phi, (\eta\mu + \frac{A^0}{\lambda_0})\phi)$ and

$$(u_1(\lambda), u_2(\lambda)) - y_0 \in W, \quad (w_1(s), w_2(s)) - y_0 \in W.$$

We will show that the hypotheses of Theorem 6.1 imply that all eigenvalues of $DF_{(A,\rho)}(\lambda_0, 0, 0)$ other than that $\sigma(0) = 0$ must be negative. The eigenvalues of $DF_{(A,\rho)}(\lambda(s), A(s), \rho(s))$ will thus be negative for $|s|$ small provided $\sigma(s) < 0$.

By assumption (1.2), we have as in [3, Theorem 1.16] that $\gamma'(\lambda_0) \neq 0$. By formula (1.17) of [3] we have that $\sigma(s)$ is not zero for all $s \neq 0$ with $|s|$ sufficiently small and $\sigma(s)$ changes sign at $s = 0$, provided $\lambda'(0) \neq 0$. In that case $\sigma(s) < 0$ either for $s > 0$ and $|s|$ sufficiently small or for $s < 0$ and $|s|$ sufficiently small.

We now proceed to calculate $\lambda'(0)$. To do so, we substitute

$$(\lambda, A, \rho) = \left(\lambda(s), s\lambda_0\phi + s\xi_1(s), s\left(\eta\mu + \frac{A^0}{\lambda_0}\right)\phi + s\xi_2(s)\right)$$

in $F(\lambda, A, \rho) = 0$, where F is as in (2.1). We differentiate the resulting equations with respect to s twice, and set $s = 0$. We obtain after some lengthy calculations that

(6.3)

$$\eta\Delta A''(0) - \frac{A^0}{\lambda_0}A''(0) + \lambda_0\rho''(0) + 2\left(\rho'(0)A'(0) + \frac{A^0}{\lambda_0^2}\lambda'(0)A'(0) + \lambda'(0)\rho'(0)\right) = 0,$$

$$\begin{aligned} &\nabla \cdot \left[\nabla\rho''(0) - \frac{2}{\lambda_0}\left(1 - \frac{A^0}{\lambda_0}\right)\nabla A''(0)\right] - \left(1 - \frac{A^0}{\lambda_0}\right)A''(0) - \lambda_0\rho''(0) \\ &+\nabla \cdot \left[\frac{4}{\lambda_0}\nabla A'(0)\left(-\rho'(0) + \left(1 - \frac{A^0}{\lambda_0}\right)\frac{A'(0)}{\lambda_0} + \frac{\lambda'(0)}{\lambda_0} - \frac{2A^0}{\lambda_0^2}\lambda'(0)\right)\right] - 2\left(\rho'(0)A'(0)\right. \\ &\left.+\frac{A^0}{\lambda_0^2}\lambda'(0)A'(0) + \lambda'(0)\rho'(0)\right) = 0. \end{aligned} \tag{6.4}$$

Simplifying (6.4), we get

$$\begin{aligned} &\frac{2}{\lambda_0}\left(1 - \frac{A^0}{\lambda_0}\right)\Delta A''(0) - \Delta\rho''(0) + \left(1 - \frac{A^0}{\lambda_0}\right)A''(0) + \lambda_0\rho''(0) \\ &-\nabla \cdot \left[\frac{4}{\lambda_0}\nabla A'(0)\left(-\rho'(0) + \left(1 - \frac{A^0}{\lambda_0}\right)\frac{A'(0)}{\lambda_0} + \frac{\lambda'(0)}{\lambda_0} - \frac{2A^0}{\lambda_0^2}\lambda'(0)\right)\right] + 2\left(\rho'(0)A'(0)\right. \\ &\left.+\frac{A^0}{\lambda_0^2}\lambda'(0)A'(0) + \lambda'(0)\rho'(0)\right) = 0. \end{aligned} \tag{6.5}$$

We next multiply (6.3) and (6.5) by ϕ and integrate over Ω. Then using integration by parts and simplifying, we see that the terms in the resulting equations involving $A''(0)$ and $\rho''(0)$ are given by

$$\begin{bmatrix} -\eta\mu - \dfrac{A^0}{\lambda_0} & \lambda_0 \\ \dfrac{2\mu}{\lambda_0}\left(1-\dfrac{A^0}{\lambda_0}\right) - \left(1-\dfrac{A^0}{\lambda_0}\right) & -\lambda_0 - \mu \end{bmatrix} \begin{bmatrix} \int_\Omega \phi A''(0) \\ \int_\Omega \phi \rho''(0) \end{bmatrix}. \tag{6.6}$$

By (3.6), the determinant of the coefficient matrix in (6.6) is zero; also $[\lambda_0 + \mu, \lambda_0]$ is a left eigenvector corresponding to the eigenvalue zero. Consequently, if we dot the resulting equations on the left by $[\lambda_0 + \mu, \lambda_0]$ and use the fact that this vector is a left eigenvector of the matrix in (6.6) corresponding to the eigenvalue zero, we obtain the following equation for $\lambda'(0)$:

(6.7)
$$2\mu \int_\Omega \phi \left(\rho'(0)A'(0) + \frac{A^0}{\lambda_0^2}\lambda'(0)A'(0) + \lambda'(0)\rho'(0)\right)$$
$$+\lambda_0 \int_\Omega \phi\nabla \cdot \left[\frac{4}{\lambda_0}\nabla A'(0)\left(-\rho'(0) + \left(1-\frac{A^0}{\lambda_0}\right)\frac{A'(0)}{\lambda_0} + \frac{\lambda'(0)}{\lambda_0} - \frac{2A^0}{\lambda_0^2}\lambda'(0)\right)\right] = 0.$$

One may check from the definition of y_0 that $A'(0) = \lambda_0\phi$ and $\rho'(0) = (\eta\mu + \frac{A^0}{\lambda_0})\phi$. Substitute these expressions into (6.7). By assumption $\int_\Omega \phi^2 = 1$. Moreover,

$$\int_\Omega \phi|\nabla\phi|^2 = \frac{1}{2}\int_\Omega \nabla\phi\nabla(\phi^2) = -\frac{1}{2}\int_\Omega \phi^2\Delta\phi = \frac{1}{2}\mu\int_\Omega \phi^3.$$

Using these facts, (6.7) reduces to

$$\left[2\left(\frac{1}{\lambda_0} - \frac{2A^0}{\lambda_0}\right) - \left(\frac{\eta\mu}{\lambda_0} + \frac{2A^0}{\lambda_0^2}\right)\right]\lambda'(0) = \left[\frac{3A^0}{\lambda_0} + 2\eta\mu - 1\right]\int_\Omega \phi^3. \tag{6.8}$$

By (3.7),

$$\frac{3A^0}{\lambda_0} + \eta\mu - 2 + \lambda_0\eta + \frac{\lambda_0}{\mu} = 0,$$

so

$$\frac{3A^0}{\lambda_0} + 2\eta\mu - 1 = \frac{1}{\mu}(\eta\mu + 1)(\mu - \lambda_0),$$

which is different from zero if $\mu \neq \lambda_0$. Hence if $\mu \neq \lambda_0$ and $\int_\Omega \phi^3 \neq 0$, it follows that $\lambda'(0) \neq 0$.

We will now verify that under suitable conditions the eigenvalue $\sigma(0) = 0$ of the linearized operator $DF_{(A,\rho)}(\lambda, A, \rho)$ at $(\lambda_0, 0, 0)$ is in fact the only nonnegative eigenvalue of $DF_{(A,\rho)}(\lambda_0, 0, 0)$, and hence is the largest eigenvalue. The eigenvalue problem has the form

$$\begin{cases} \eta\Delta u - \dfrac{A^0}{\lambda}u + \lambda v = \sigma u \quad \text{in} \quad \Omega, \\ \dfrac{2}{\lambda}\left(1 - \dfrac{A^0}{\lambda}\right)\Delta u - \Delta v + \left(1 - \dfrac{A^0}{\lambda}\right)u + \lambda v = \sigma v \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{6.9}$$

Suppose that A and B are 2×2 matrices and $\vec{w}$ is a vector valued function such that

$$A\Delta\vec{w} + B\vec{w} = \sigma\vec{w}. \tag{6.10}$$

Recall that $\vec{w}$ has a componentwise expansion in orthonormal eigenfunctions ϕ_k of $-\Delta$ with Neumann boundary conditions as $\vec{w} = \Sigma_{k=0}^{\infty}\phi_k\vec{w}_k$, and substitute that expansion into (6.10). Let μ_k denote the eigenvalue for ϕ_k. If σ is an eigenvalue of (6.10), then $\vec{w}_i \neq 0$ for some i. Multiplying (6.10) by ϕ_i, integrating over Ω, and using orthogonality yields $\sigma\vec{w}_i = -A\mu_i\vec{w}_i + B\vec{w}_i$. It follows that any eigenvalue σ of (6.10) can be expressed as an eigenvalue of the matrix $-A\mu_i + B$ for some eigenvalue μ_i of $-\Delta$. Conversely, any eigenvalue μ_i of $-\Delta$ will lead to two (not necessarily distinct) eigenvalues of (6.10) corresponding to the eigenvalues of $-A\mu_i + B$. The matrix equation determining such eigenvalues for (6.9) is

$$\begin{bmatrix} -\eta\mu - \dfrac{A^0}{\lambda} & \lambda \\ \dfrac{2\mu}{\lambda}\left(1 - \dfrac{A^0}{\lambda}\right) - \left(1 - \dfrac{A^0}{\lambda}\right) & -\lambda - \mu \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \sigma \begin{bmatrix} U \\ V \end{bmatrix}. \tag{6.11}$$

To find σ we compute the determinant of the matrix obtained by subtracting σI from the matrix on the left of (6.11) and set it equal to zero. After simplification that yields

$$\sigma^2 + \left(\eta\mu + \frac{A^0}{\lambda} + \lambda + \mu\right)\sigma + C(\mu, \lambda) = 0, \tag{6.12}$$

where

$$\begin{aligned} C(\mu, \lambda) &= \mu\lambda\left[3\frac{A^0}{\lambda^2} + \frac{(\eta\mu - 2)}{\lambda} + \eta + \frac{1}{\mu}\right] \\ &= \eta\mu^2 + \left(3\frac{A^0}{\lambda} - 2 + \lambda\eta\right)\mu + \lambda. \end{aligned} \tag{6.13}$$

At the bifurcation point $(\lambda(0), A(0), \rho(0)) = (\lambda_0, 0, 0)$ we have $\sigma = 0$ as an eigenvalue of (6.11), so if μ_j is the associated eigenvalue of $-\Delta$, then we have $C(\mu_j, \lambda_0) = 0$, which recovers (3.7). The second eigenvalue σ associated with μ_j is negative. If we think of $C(\mu, \lambda_0) = 0$ as a quadratic in μ, then we can compute the second root μ_j^* by factoring $\mu - \mu_j$ out of the equation $C(\mu, \lambda_0) - C(\mu_j, \lambda_0) = 0$. That yields

$$\mu_j^* = \frac{2}{\eta} - 3\frac{A^0}{\eta\lambda_0} - \lambda_0 - \mu_j = \lambda_0 + \frac{\lambda_0}{\mu\eta}, \tag{6.14}$$

where the last quantity follows from (3.7). Clearly $\mu_j^* \in (0, \frac{2}{\eta})$. (Note that in general μ_j^* is not an eigenvalue of $-\triangle$.) If there is no eigenvalue $\mu_k \neq \mu_j$ of $-\triangle$ lying in the closed interval with endpoints μ_j and μ_j^*, then all eigenvalues σ of $DF_{(A,\rho)}(\lambda_0, 0, 0)$ arising from eigenvalues of $-\triangle$ other than μ_j will be negative, so that $\sigma(0) = 0$ is the largest eigenvalue of $DF_{(A,\rho)}(\lambda_0, 0, 0)$.

The closed interval with endpoints μ_j and μ_j^* is contained in the interval $(0, \frac{2}{\eta})$, so in particular this will be the case if there are no eigenvalues of $-\triangle$ in $(0, \frac{2}{\eta})$ other than μ_j.

7. Conclusions. We have established that under suitable conditions on the parameters, branches of spatially heterogeneous solutions to the model (EQ) bifurcate from the spatially constant equilibrium. The bifurcations are global, so roughly speaking the solution branches must either connect with each other, persist for an unbounded set of parameter values, or the solutions must grow in such a way that they leave the region in state space where the bifurcation analysis is formulated. In any of those cases the bifurcating branches will include solutions at some distance from the points where they bifurcate. Near a bifurcation point the spatial variation in the solutions will be close to that of the eigenfunction of the Laplacian associated with that bifurcation point, so that peaks in that eigenfunction would correspond to hotspots. However, the pattern may change as the branch moves away from the bifurcation point. Moreover, under additional assumptions on the geometry of the underlying spatial domain, we have shown that near the bifurcation point, some of these branches include solutions that are stable when viewed as equilibria to the corresponding time dependent problem. The geometry of the domain enters into the conditions for stability in terms of hypotheses on the eigenvalues and eigenfunctions of the Laplace operator with Neumann boundary conditions. Specifically, these conditions are satisfied in some cases for radially symmetric solutions on a disk. In that case the bifurcation is transcritical. In the cases where the underlying domain is an interval or rectangle, the bifurcation is "vertical." These observations are consistent with those obtained by asymptotic analysis in [11].

The analysis in the present paper compliments and in some ways expands upon that of [11]. On the mathematical side, our results are mathematically rigorous, as opposed to the formal perturbation analysis in [11]. They include global bifurcation results as well as a local bifurcation and stability analysis. Our methods allow us to treat the case of no-flux boundary conditions on general domains. That extends the scope of the analysis beyond that of [11], which is based on case by case computations that use the explicit formulas for eigenfunctions in specific geometries. (The results in [11] treat radially symmetric solutions on a disk with no-flux boundary conditions and spatially periodic solutions in one dimension or in two dimensions in the cases of symmetries arising from periodicity relative to square or hexagonal tilings of the plane.) On the applied side, the fact that we consider no-flux boundary conditions on general domains means that our results imply relationships between the size and shape of the underlying spatial region and the nature and stability of the spatial patterns that the models can support. The bifurcation points for the model are determined by eigenvalues of the Laplacian under Neumann boundary conditions. That observation provides a starting point for developing a "sociogeographic" description of crime patterns analogous to biogeographic descriptions of ecological communities. As noted previously, near a bifurcation point, the spatial pattern generated by the model is approximately that of the relevant eigenfunction of the Laplacian. For pattern formation in general, only the nonzero eigenvalues are relevant because they have nonconstant eigenfunctions. By condition (1.2), the only relevant eigenvalues for this model are those that are sufficiently small. Domains of a given shape can be parameterized as $\Omega_\ell = \ell\Omega_0$, where Ω_0 is some specified domain and ℓ represents a linear scale factor. On such a family of domains, the nonzero eigenvalues of the Laplacian scale as $1/\ell^2$, so larger domains will typically have higher eigenvalues that still satisfy the first inequality of (1.2). Since the eigenfunctions associated with higher eigenvalues typically have more complicated spatial patterns than those associated with lower eigenvalues, it thus would be expected that for larger regions the model could potentially support more complicated patterns of criminal activity, which for at least some parameter ranges would be described by the patterns of the relevant eigenfunctions. Thus, our results provide a rigorous mathematical framework that allows a broader and deeper study of the patterns of burglary predicted by the model, particularly as they relate to the size and shape of the spatial region upon which the model is formulated.

Acknowledgment. We thank two anonymous referees for suggestions that improved the paper.

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